

Radiative decays of 1^{++} heavy mesons in the covariant light-front approach

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We calculate the predicted width for the radiative decay of a 1^{++} heavy meson via the channel $1^{++} \rightarrow 1^{--} + \gamma$ in the covariant light-front quark model. Specifically, we compute the decay widths for $\chi_{c1}(1P) \rightarrow J/\psi + \gamma$ and $\chi_{b1}(nP) \rightarrow \Upsilon(n'S) + \gamma$. The results are compared with experimental data and with predictions from calculations based on nonrelativistic models and their extensions to include relativistic effects.

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I. INTRODUCTION

Experimental observations and theoretical studies of heavy quarkonium $Q\bar{Q}$ states have played a very valuable role in elucidating the properties of quantum chromodynamics (QCD). A heavy quark is one whose mass, m_Q , is large compared with $\Lambda_{QCD} \sim 0.3$, so that the running QCD coupling $g_s(M_Q)$ and the associated quantity $\alpha_s(m_Q) = g_s(M_Q)^2/(4\pi)$ are reasonably small, allowing perturbative treatments of at least some parts of the physics of $Q\bar{Q}$ states and decays. Furthermore, for $m_Q \gg \Lambda_{QCD}$, one can obtain an approximate description of many properties of the $Q\bar{Q}$ states using nonrelativistic methods, including potential models. From the time of the discovery of the J/ψ at BNL [1] and SLAC/SPEAR [2] in 1974, and the Υ at Fermilab in 1977 [3, 4], there has been a steadily growing wealth of data on the various $Q\bar{Q}$ states, where the Q denotes a charm quark c or a bottom/beauty quark b , as well as data on mesons and baryons with charm and bottom/beauty quantum numbers. Some reviews of heavy quarkonia and references to the literature include [5]-[15].

These experimental achievements motivate the continued theoretical study of the structure and properties of $c\bar{c}$ and $b\bar{b}$ quarkonium states. Among quarkonium decays, radiative decays are particularly valuable as tests of various models, since the photon is directly observed and the nature of the electromagnetic transition is well understood. One of the simplest types of radiative decays is the electric dipole (E1) transition between a $Q\bar{Q}$ state with radial quantum number n and spectroscopic type $n^{2S+1}L_J = n^3L_J$ with $L = 1$ (P-wave) and $J = 0, 1, 2$, denoted $\chi_{QJ}(nP)$ in standard notation, where $Q = c, b$, and a lower-lying $Q\bar{Q}$ S-wave state n'^3S_1 , in particular, J/ψ and $\psi(2S)$ for the $c\bar{c}$ system, and $\Upsilon(n'S)$ with $n' = 1, 2, 3$ for the $b\bar{b}$ system. In terms of J^{PC} values, these decays are of the form $J^{++} \rightarrow 1^{--} + \gamma$, where $J = 0, 1, 2$. The P-wave $c\bar{c}$ states were first observed in 1976 by the SLAC-LBL experiment at SLAC/SPEAR [16, 17]. The P-wave $b\bar{b}$ states were first observed by the Columbia-Stony Brook (CUSB) experiment at the Cornell CESR e^+e^- storage ring [18, 19] and confirmed by the CLEO experiment at CESR [20]. Larger data samples and quite accurate measurements of branching ratios for radiative decays of P-wave $b\bar{b}$ states were obtained

later, in particular, by the CLEO III [21] and BABAR experiments [22]. Valuable results have also been obtained from hadron colliders, including the observation of the $\chi_{bJ}(3P) b\bar{b}$ states at the Large Hadron Collider (LHC) [23] and the measurement of the mass of $\chi_{b1}(3P)$ by LHCb [24].

There have been a number of theoretical studies of these E1 transitions based on a range of different models [25]-[33]. Many of these models make use of nonrelativistic potentials, such as the potential $V = -(4/3)\alpha_s(m_Q)/r + \sigma r$, where the first term is a non-Abelian Coulomb potential representing one-gluon exchange at short distances and the second term is the linear confining potential, with σ denoting the string tension. These are reasonable models, since a $Q\bar{Q}$ bound-state system is nonrelativistic if $m_Q/\Lambda_{QCD} \gg 1$.

It is of interest to investigate these radiative decays of P-wave quarkonium states using a different type of model, namely the light-front quark model (LFQM)[34]-[44]. This approach permits a fully relativistic treatment of the quark spins and the internal motion of the constituent quarks. In this covariant approach, the hadronic structure for small momentum transfer is represented by one-loop diagrams evaluated on the light cone. It has been used to study semileptonic and nonleptonic decays of heavy-flavor D and B mesons and also to evaluate radiative decay rates of heavy mesons [45-49]. In particular, in [49] with Ke and Li, we used this approach to calculate the widths for the radiative decays of heavy 0^{++} and 1^{+-} $Q\bar{Q}$ mesons.

In the light-front formalism, one chooses the coordinate where $q^+ = 0$, in which the quark current cannot create or annihilate pairs, and the relevant transition matrix element can be computed as an overlap of Fock-space wavefunctions. The terms involving pair production or annihilation vanish [38, 50]. An advantage of the light-front quark model is that it is manifestly covariant. In the light-front approach, it is easy to boost a hadron bound states from one inertial Lorentz frame to another one when the bound state wavefunction is known in a particular frame[38].

In this paper, extending our previous work with Ke and Li in Ref. [49], we study the radiative decays

$$\chi_{c1}(1P) \rightarrow J/\psi + \gamma \quad (1.1)$$

and

$$\chi_{b1}(nP) \rightarrow \Upsilon(n'S) + \gamma \quad (1.2)$$

where $n \geq n'$ by using the light-front quark model. With the front-front formalism, we perform a numerical calculation the widths for these decays and then compare the results with theoretical calculations based on other approaches.

The paper is organized as follows: In Section II, we derive the formulas for the radiative decay $1^{++} \rightarrow 1^{--} + \gamma$. Then in section III, we discuss the meson wavefunctions that are relevant to the light-front approach. In Section IV, we discuss numerical results for the decay widths of $\chi_{c1}(1P) \rightarrow J/\psi + \gamma$ and $\chi_{b1}(nP) \rightarrow \Upsilon(n'S) + \gamma$. Our conclusions are given in Section V.

II. LIGHT-FRONT FORMALISM FOR THE DECAYS $1^{++} \rightarrow 1^{--} + \gamma$

A. Notation

Here we briefly summarize the notation that is relevant for radiative transition of meson. We follow the covariant light-front approach of [42, 44] and use the same notation. In light-front coordinates, a (four-)momentum p is expressed as

$$p^\mu = (p^-, p^+, \vec{p}_\perp) \quad (2.1)$$

where

$$p^+ = p^0 + p^3, \quad p^- = p^0 - p^3 \quad (2.2)$$

and

$$\vec{p}_\perp = (p^1, p^2). \quad (2.3)$$

Thus,

$$\begin{aligned} p^2 &= (p^0)^2 - |\vec{p}|^2 = (p^0)^2 - (p^3)^2 - |\vec{p}_\perp|^2 \\ &= p^+ p^- - |\vec{p}_\perp|^2. \end{aligned} \quad (2.4)$$

We denote the momentum of the parent (incoming) meson as $P' = p'_1 + p_2$, where p'_1 and p_2 are the momenta of the constituent quark and anti-quark, with mass m'_1 and m_2 , respectively. Similarly, we label the momentum of the daughter (outgoing) quarkonium meson as $P'' = p''_1 + p_2$, where p''_1 is the momentum of the constituent quark, with mass m'_1 . For our application to $Q\bar{Q}$ quarkonium systems, $m'_1 = m_2 = m'' = m_Q$. The four-momentum of the parent meson with mass M' , in terms of light-front coordinates, is

$$P' = (P'^-, P'^+, \vec{P}'_\perp) \quad (2.5)$$

so $P'^2 = P'^+ P'^- - |\vec{P}'_\perp|^2 = M'^2$. Similarly, for the outgoing meson, $P''^2 = M''^2$. In what follows, the vector

signs on transverse momentum components are to be understood implicitly and are suppressed in the notation. The internal motion of the constituents can be described by the variables (x_2, p'_\perp) , where

$$\begin{aligned} p_1^+ &= x_1 P'^+, \quad p_2^+ = x_2 P'^+ \\ p_{1\perp} &= x_1 P'_\perp + p'_\perp, \quad p_{2\perp} = x_2 P'_\perp - p'_\perp \\ x_1 + x_2 &= 1 \end{aligned} \quad (2.6)$$

and p''_\perp can be expressed as

$$p''_\perp = p'_\perp - x_2 q_\perp. \quad (2.7)$$

B. Form factors

Let us define $P = P' + P''$ and $q = P' - P''$. Since the initial P-wave $1^{++} Q\bar{Q}$ state is an axial-vector, we denote it as A , while the final $1^{--} Q\bar{Q}$ state is a vector, denoted V . The general amplitude for the transition $A(1^{++}) \rightarrow V(1^{--}) + \gamma$ has the form

$$i\mathcal{A}(A(P') \rightarrow V(P'')\gamma(q)) = \varepsilon_\mu^*(q) \varepsilon'_\nu(P') \varepsilon''^*_\rho(P'') i\tilde{\mathcal{A}}^{\mu\nu\rho}, \quad (2.8)$$

where $\varepsilon'_\nu(P')$, $\varepsilon''^*_\rho(P'')$, and $\varepsilon_\mu^*(q)$ are the polarization (four-)vectors of the parent heavy axial-vector meson, the daughter heavy vector meson, and the photon, respectively. The structure of this amplitude was given in [51]. We review this next. Since quantum electrodynamic (QED) interactions are invariant under parity and time reversal (and thus also CP), this amplitude must be P- and T-invariant. In addition to these two conditions, the transverse properties of the polarization vectors yield the three further conditions

$$\varepsilon'_\nu(P')(P + q)^\nu = 0, \quad (2.9)$$

$$\varepsilon''^*_\rho(P'')(P - q)^\rho = 0, \quad (2.10)$$

and

$$\varepsilon_\mu^*(q)q^\mu = 0. \quad (2.11)$$

Condition (2.11) is also implied by electromagnetic gauge invariance. Applying these conditions, we obtain the following general amplitude (to be simplified below):

$$\begin{aligned} i\tilde{\mathcal{A}}^{\mu\nu\rho} &= f_1 \epsilon^{\mu\nu\rho\alpha} P_\alpha + f_2 \epsilon^{\mu\nu\rho\alpha} q_\alpha + f_3 \epsilon^{\rho\nu\alpha\beta} P^\mu P_\alpha q_\beta \\ &\quad + f_4 \epsilon^{\mu\nu\alpha\beta} P^\rho P_\alpha q_\beta + f_5 \epsilon^{\rho\mu\alpha\beta} P^\nu P_\alpha q_\beta. \end{aligned} \quad (2.12)$$

This expression can be simplified by using the fact that the photon only has two transverse polarization states, so the timelike component $\varepsilon_0^*(q) = 0$. Taking the parent axial vector meson $A(P')$ to be in its rest frame, we have $P'^\mu = (M', 0)$, where M' is mass of $A(P')$. The f_3 term can be eliminated:

$$\begin{aligned} [f_3 \epsilon^{\rho\nu\alpha\beta} P^\mu P_\alpha q_\beta] \varepsilon_\mu^*(q) \varepsilon'_\nu(P') \varepsilon''^*_\rho(P'') &\propto \varepsilon_\mu^*(q) P^\mu \\ &= 2\varepsilon_\mu^*(q) P'^\mu = 2\varepsilon_0^*(q) P'^0 = 0. \end{aligned} \quad (2.13)$$

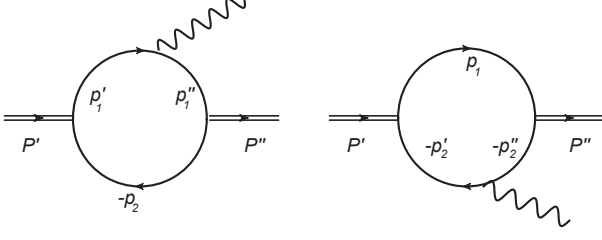


FIG. 1: Feynman diagrams for radiative transitions in the light-front framework.

Furthermore, the f_1 term vanishes due to electromagnetic gauge invariance, $q_\mu \tilde{A}^{\mu\nu\rho} = 0$. Therefore, the general amplitude that satisfies the five conditions above is given by [51]:

$$\begin{aligned} i\tilde{A}^{\mu\nu\rho} &\rightarrow iA^{\mu\nu\rho} \\ &= f_2 \epsilon^{\mu\nu\rho\alpha} q_\alpha + f_4 \epsilon^{\mu\nu\alpha\beta} P^\rho P_\alpha q_\beta + f_5 \epsilon^{\rho\mu\alpha\beta} P^\nu P_\alpha q_\beta. \end{aligned} \quad (2.14)$$

The f_2 term corresponds to the electric dipole (E1) transition and makes the dominant contribution to the amplitude, while the f_4 and f_5 terms correspond to the magnetic dipole (M2) transition and make subdominant contributions [52, 53]. A detailed analysis of parity and time-reversal invariance of this general amplitude is given in Appendix A.

C. Calculation of radiative decay amplitude

In general, the width for an electromagnetic dipole transition between an initial $Q\bar{Q}$ state n^3P_J and a final state $n'^3S_1 + \gamma$ is given by (e.g., [8])

$$\Gamma(n^3P_J \rightarrow n'^3S_1 + \gamma) = \frac{4\alpha_{em} e_Q^2 E_\gamma^3}{9} |\langle f | \vec{r} | i \rangle|^2, \quad (2.15)$$

where e_Q is the quark of the quark Q , E_γ is the energy of the outgoing photon in the parent rest frame, and i and j denote the initial- and final-state wavefunctions. For our calculation in the LFQM, we note that the vertex function for the parent axial-vector meson $A(1^{++})$ is given by [44]:

$$iH'_A \left[\gamma^\nu + \frac{(p'_1 - p_2)^\nu}{W'_A} \right] \gamma^5, \quad (2.16)$$

and the vertex function for the daughter vector meson $V(1^{--})$ is

$$iH''_V \left[\gamma^\rho - \frac{(p''_1 - p_2)^\rho}{W''_V} \right], \quad (2.17)$$

where H'_A and H''_V are functions of p'_1 , p''_1 and p_2 . The explicit forms for these vertex functions will be discussed below.

There are two diagrams that contribute at leading order to the $A \rightarrow V + \gamma$ transition amplitude, so we write

$$i\mathcal{A}^{\mu\nu\rho}(A \rightarrow V + \gamma) = i\mathcal{A}^{\mu\nu\rho}(a) + i\mathcal{A}^{\mu\nu\rho}(b), \quad (2.18)$$

where the left-hand diagram in Fig.(1) corresponds to $\mathcal{A}^{\mu\nu\rho}(a)$ and the right-hand diagram in Fig.(1) corresponds to $\mathcal{A}^{\mu\nu\rho}(b)$. These are related by charge conjugation. For the left-hand diagram, the transition amplitude is given by

$$i\mathcal{A}^{\mu\nu\rho}(a) = eN_{e1'} \frac{N_c}{(2\pi)^4} \int d^4 p'_1 \frac{H'_A H''_V}{N'_1 N''_1 N_2} \mathcal{S}_a^{\mu\nu\rho}, \quad (2.19)$$

where

$$\begin{aligned} \mathcal{S}_a^{\mu\nu\rho} &= \text{Tr} \left\{ (p''_1 + m''_1) \gamma^\mu (p'_1 + m'_1) \left[\gamma^\nu + \frac{(p'_1 - p_2)^\nu}{W'_A} \right] \right. \\ &\quad \times \left. \gamma^5 (-\not{p}_2 + m_2) [\gamma^\rho - \frac{1}{W''_V} (p''_1 - p_2)^\rho] \right\} \\ &= -p_1^{\mu\nu} \text{Tr} [\not{p}'_1 \gamma^\nu \not{p}_2 \gamma^\rho \gamma^5] + (p'_1 \cdot p''_1 - m'_1 m''_1) \\ &\quad \times \text{Tr} [\gamma^\mu \gamma^\nu \not{p}_2 \gamma^\rho \gamma^5] - p_1^{\mu\nu} \text{Tr} [\not{p}'_1 \gamma^\nu \not{p}_2 \gamma^\rho \gamma^5] \\ &\quad - \frac{m''_1}{W'_A} (p'_1 - p_2)^\nu \text{Tr} [\gamma^\mu \not{p}'_1 \not{p}_2 \gamma^\rho \gamma^5] - m''_1 m_2 \\ &\quad \times \text{Tr} [\gamma^\mu \not{p}'_1 \gamma^\nu \gamma^\rho \gamma^5] - \frac{m'_1}{W_A} (p'_1 - p_2)^\nu \text{Tr} [\not{p}'_1 \gamma^\mu \not{p}_2 \gamma^\rho \gamma^5] \\ &\quad - m'_1 m_2 \text{Tr} [\not{p}'_1 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^5] - \frac{m'_1}{W''_V} (p''_1 - p_2)^\rho \\ &\quad \times \text{Tr} [\not{p}'_1 \gamma^\mu \gamma^\nu \not{p}_2 \gamma^5] - \frac{m''_1}{W''_V} (p''_1 - p_2)^\rho \text{Tr} [\gamma^\mu \not{p}'_1 \gamma^\nu \not{p}_2 \gamma^5] \\ &\quad - \left[\frac{(p'_1 - p_2)^\nu}{W'_A} m_2 + p_2^\nu \right] \text{Tr} [\not{p}'_1 \gamma^\mu \not{p}'_1 \gamma^\rho \gamma^5] \\ &\quad - \left[\frac{(p'_1 - p_2)^\nu}{W'_A} \frac{(p''_1 - p_2)^\rho}{W''_V} - g^{\rho\nu} \right] \text{Tr} [\not{p}'_1 \gamma^\mu \not{p}'_1 \not{p}_2 \gamma^5] \\ &\quad - \left[p_2^\rho + \frac{m_2}{W''_V} (p''_1 - p_2)^\rho \right] \text{Tr} [\not{p}'_1 \gamma^\mu \not{p}'_1 \gamma^\nu \gamma^5], \end{aligned} \quad (2.20)$$

$$N'_1 = p_1'^2 - m_1'^2 + i\epsilon, \quad (2.21)$$

$$N''_1 = p_1''^2 - m_1''^2 + i\epsilon, \quad (2.22)$$

$$N_2 = p_2^2 - m_2^2 + i\epsilon, \quad (2.23)$$

and $N_{e1'(\epsilon_2)}$ denotes the electric charge of the constituent quark in units of e . The contribution to the amplitude from the right-hand diagram follows from this.

To calculate the amplitude in the covariant light-front approach, we need to integrate over the internal momentum, p'_1 . In order to do this, we first express the amplitude in terms of internal momentum p'_1 and external momenta P and q , as well as N'_1 , N''_1 , N_2 , by using the following relations:

$$p''_1 = p'_1 - q$$

$$\begin{aligned}
p_2 &= (P + q)/2 - p'_1 \\
2p'_1 \cdot p_2 &= M'^2 - N_1' - m_1'^2 - N_2 - m_2^2 \\
2p_1'' \cdot p_2 &= M''^2 - N_1'' - m_1''^2 - N_2 - m_2^2 \\
2p'_1 \cdot p_1'' &= -q^2 + N_1' + m_1'^2 + N_1'' + m_1''^2. \quad (2.24)
\end{aligned}$$

After the integration over $p_1'^-$, one makes the following replacement [42, 44]:

$$\begin{aligned}
&\int d^4 p'_1 \frac{H'_A H''_V}{N'_1 N''_1 N_2} S_a^{\mu\nu\rho} \varepsilon_\mu^* \varepsilon'_\nu \varepsilon''_\rho \rightarrow \\
&-i\pi \int dx_2 d^2 p'_\perp \frac{h'_A h''_V}{x_2 \hat{N}'_1 \hat{N}''_1} \hat{S}_a^{\mu\nu\rho} \hat{\varepsilon}_\mu^* \hat{\varepsilon}'_\nu \hat{\varepsilon}''_\rho, \quad (2.25)
\end{aligned}$$

where

$$\begin{aligned}
N_1' &\rightarrow \hat{N}_1' = x_1(M'^2 - M_0'^2) \\
N_1'' &\rightarrow \hat{N}_1'' = x_1(M''^2 - M_0''^2) \\
H'_A &\rightarrow h'_A = (M'^2 - M_0'^2) \sqrt{\frac{x_1 x_2}{N_c}} \frac{\tilde{M}'_0}{4M'_0} \phi_{np}(x_2, p'_\perp) \\
H''_V &\rightarrow h''_V = (M''^2 - M_0''^2) \sqrt{\frac{x_1 x_2}{N_c}} \frac{1}{\sqrt{2}\tilde{M}_0''} \phi_{n's}(x_2, p''_\perp) \\
W'_A &\rightarrow w'_A = \frac{\tilde{M}_0'^2}{m'_1 - m_2} \\
W''_V &\rightarrow w''_V = M_0'' + m_1'' + m_2. \quad (2.26)
\end{aligned}$$

In the above expressions, $\phi_{n's}$ and ϕ_{np} represent the wavefunction for the S-wave $Q\bar{Q}$ meson V and the P-

wave $Q\bar{Q}$ meson A , respectively. We will discuss these wavefunctions in detail in the next section. The definitions of M'_0 , M''_0 , \tilde{M}'_0 and \tilde{M}''_0 are given in appendix C. The definition of $\hat{\varepsilon}^*$, $\hat{\varepsilon}'$ and $\hat{\varepsilon}''^*$ is given in [42, 44].

One should also include the contribution from zero modes in the A meson. In practice, this amounts to the following replacement for $p_{1\mu}'$ in $\hat{S}_a^{\mu\nu\rho}$ in the integral [42, 44]:

$$\begin{aligned}
\hat{p}'_{1\mu} &\rightarrow P_\mu A_1^{(1)} + q_\mu A_2^{(1)}, \\
\hat{p}'_{1\mu} \hat{p}'_{1\nu} &\rightarrow g_{\mu\nu} A_1^{(2)} + P_\mu P_\nu A_2^{(2)} \\
&\quad + (P_\mu q_\nu + q_\mu P_\nu) A_3^{(2)} + q_\mu q_\nu A_4^{(2)}, \\
\hat{p}'_{1\mu} \hat{p}'_{1\nu} \hat{p}'_{1\alpha} &\rightarrow (g_{\mu\nu} P_\alpha + g_{\mu\alpha} P_\nu + g_{\nu\alpha} P_\mu) A_1^{(3)} \\
&\quad + (g_{\mu\nu} q_\alpha + g_{\mu\alpha} q_\nu + g_{\nu\alpha} q_\mu) A_2^{(3)} \\
&\quad + P_\mu P_\nu P_\alpha A_3^{(3)} + (P_\mu P_\nu q_\alpha + P_\mu q_\nu P_\alpha \\
&\quad + q_\mu P_\nu P_\alpha) A_4^{(3)} + (q_\mu q_\nu P_\alpha + q_\mu P_\nu q_\alpha \\
&\quad + P_\mu q_\nu q_\alpha) A_5^{(3)} + q_\mu q_\nu q_\alpha A_6^{(3)}. \quad (2.27)
\end{aligned}$$

After these operations, the amplitude $\mathcal{A}^{\mu\nu\rho}(a)$ can be expressed as a function of the external four-momenta P and q . It can be parametrized in the following form:

$$\begin{aligned}
i\mathcal{A}^{\mu\nu\rho}(a) &= f_2^a \epsilon^{\mu\nu\rho\alpha} q_\alpha + f_4^a \epsilon^{\mu\nu\alpha\beta} P^\rho P_\alpha q_\beta \\
&\quad + f_5^a \epsilon^{\rho\mu\alpha\beta} P^\nu P_\alpha q_\beta, \quad (2.28)
\end{aligned}$$

with

$$\begin{aligned}
f_2^a(q^2) &= eN_{e_1} \frac{N_c}{16\pi^3} \int dx_2 d^2 p'_\perp \frac{h'_A h''_V}{x_2 \hat{N}'_1 \hat{N}''_1} (-4) \cdot \left[\frac{1}{w'_A} (m_1'' + m_1' - 2m_2) A_1^{(2)} + \frac{1}{w''_V} (2m_2 + m_1' + m_1'') A_1^{(2)} \right. \\
&\quad \left. - \frac{1}{4} (1 - 2A_2^{(1)}) (-q^2 + \hat{N}_1' + \hat{N}_1'' + (m_1' - m_1'')^2) - A_2^{(1)} (m_1'' m_2 - m_1' m_2) - m_1' m_2 \right], \\
f_4^a(q^2) &= eN_{e_1} \frac{N_c}{16\pi^3} \int dx_2 d^2 p'_\perp \frac{h'_A h''_V}{x_2 \hat{N}'_1 \hat{N}''_1} (-4) \cdot \left[\frac{1}{w'_V} \left((m_1' - m_1'') (A_3^{(2)} + A_4^{(2)} - A_2^{(1)}) + (m_1' + m_1'' + 2m_2) \right. \right. \\
&\quad \left. \left. \times (A_2^{(2)} + A_3^{(2)} - A_1^{(1)}) - m_1' (A_1^{(1)} + A_2^{(1)} - 1) \right) - A_1^{(1)} + A_2^{(2)} + A_3^{(2)} - \frac{1}{w'_A w''_V} (2A_1^{(3)} + 2A_2^{(3)} - 2A_1^{(2)}) \right] \\
f_5^a(q^2) &= eN_{e_1} \frac{N_c}{16\pi^3} \int dx_2 d^2 p'_\perp \frac{h'_A h''_V}{x_2 \hat{N}'_1 \hat{N}''_1} (-4) \cdot \left[\frac{1}{w'_A} \left((m_1' - m_1'') (A_3^{(2)} - A_4^{(2)}) + (m_1' + m_1'' - 2m_2) \right. \right. \\
&\quad \left. \left. \times (A_2^{(2)} - A_3^{(2)}) + m_1' (A_2^{(1)} - A_1^{(1)}) \right) + A_2^{(2)} - A_3^{(2)} - \frac{1}{w'_A w''_V} (2A_2^{(3)} - 2A_1^{(3)}) \right], \quad (2.29)
\end{aligned}$$

where the explicit expression of $A_j^{(i)}$ is given in Appendix C.

For the right-hand diagram in Fig.(1), the amplitude $\mathcal{A}^{\mu\nu\rho}(b)$ can be obtained from $\mathcal{A}^{\mu\nu\rho}(a)$ by the interchanges $m_1' \leftrightarrow m_2'$, $m_1'' \leftrightarrow m_2''$, $m_2 \leftrightarrow m_1$, $N_{e_1} \leftrightarrow N_{e_2}$:

$$i\mathcal{A}^{\mu\nu\rho}(b) = f_2^b \epsilon^{\mu\nu\rho\alpha} q_\alpha + f_4^b \epsilon^{\mu\nu\alpha\beta} P^\rho P_\alpha q_\beta$$

$$+ f_5^b \epsilon^{\rho\mu\alpha\beta} P^\nu P_\alpha q_\beta. \quad (2.30)$$

The coefficients f_i in Eq.(2.14) are the sum of contributions from two parts, $\mathcal{A}^{\mu\nu\rho}(a)$ and $\mathcal{A}^{\mu\nu\rho}(b)$:

$$f_i(q^2) = f_i^a(q^2) + f_i^b(q^2), \quad (i = 2, 4, 5). \quad (2.31)$$

In Eqs. (2.29) and (2.31) we write these as general form

TABLE I: Decay width (in units of keV) of $\chi_{c1}(1P) \rightarrow J/\psi + \gamma$ in the light-front approach, based on modified harmonic oscillator wavefunctions [48]. The predictions from other models (relativistic quark model[29, 31], nonrelativistic screened potential model [9]) and experimental data from PDG [71] are also listed for comparison. The parameters sets are CM1 and CM2. We use the PDG fitted value $\Gamma_{\chi_{c1}} = 840 \pm 40$ keV and $BR(\chi_{c1}(1P) \rightarrow J/\psi + \gamma) = 33.9 \pm 1.2$ % [71]. For the entry referring to Ref. [29], we list three values presented there, based on the specific models used in that work.

Decay mode	CM1	CM2	exp.(PDG)[71]	[9](NR)	[29]
$\chi_{c1}(1P) \rightarrow J/\psi + \gamma$	324 ± 20	282 ± 35	285 ± 30	241	265/285/305

factors dependent on q^2 , but note that in the physical $A \rightarrow V\gamma$ decay, $q^2 = 0$ for the real outgoing photon, so that these are simply constant coefficients. We use this generalization to nonzero q^2 because in the light-front formalism, these form factors are calculated in the region

where the photon momentum is not onshell, i.e., where $q^2 \neq 0$. To obtain the physical values $f_i(0)$ and calculate the decay rate, we take limit $q^2 \rightarrow 0$. This yields the resulting width

$$\Gamma(n^3P_1 \rightarrow n'^3S_1) = \frac{q^3}{24\pi} \left\{ \frac{2}{M'^2} [f_2^2 + 4f_2f_4M'q + 4M'^2q^2f_4^2] + \frac{2}{M'^2} [f_2^2 - 4f_2f_5M'q + 4M'^2q^2f_5^2] \right\}, \quad (2.32)$$

where $q = (M'^2 - M'^2)/(2M')$ is the momentum of the emitted photon. In this paper we focus on E1 dipole transition rates, which are dominant, and hence drop the subdominant f_4 and f_5 terms in the calculations.

III. WAVEFUNCTIONS FOR HEAVY QUARKONIUM STATES

The wavefunctions $\phi_{n's}$ and ϕ_{np} can, in principle, be derived from relativistic light-front Bethe-Salpeter type equations [41, 54]. However, as discussed in Refs. [41] and [55], there is a simpler approach, namely to use wavefunctions from nonrelativistic quark models with given potentials. Although a QCD-motivated potential has the form $V = -(4/3)\alpha_s(m_Q)/r + \sigma r$, as noted above, this involves the complication of requiring numerical solutions of the Schrödinger equation. To avoid this complication, Refs. [41] and [55] used variational solutions of the Schrödinger equation with a nonrelativistic harmonic oscillator potential. This approach was also adopted by Refs. [42, 44, 56, 57]. However, the predictions from this type of approach do not fit the measured widths for $\Upsilon(nS)$ well, and to overcome this problem, modified harmonic oscillator wavefunctions were suggested in [48]. The normalization and explicit expressions for the modified harmonic wavefunctions are listed in Appendix B.

In the next section, we use the modified harmonic oscillator wavefunctions in [48] to calculate numerically the

radiative decay widths of $\chi_{c1}(1P)$ and $\chi_{b1}(nP)$ states and to compare these with theoretical predictions from other models.

Some comments are appropriate concerning approaches other than the light-front approach. For the heavy quarkonium system, nonrelativistic potential models such as Cornell potential model have proved to be generally rather successful in fitting data [58–62]. There are also analyses of relativistic corrections to potential models, such as [26, 63, 64]. A relativistic quark model was proposed in Ref. [65]. Screening effects were studied in Refs. [66–68], and additional potential models were used in [69, 70]. In these potential models, the wavefunctions can be obtained by numerically solving the Schrödinger equations. In future work it would be of interest to investigate the differences in radiative widths calculated using the phenomenological wavefunctions for the light-front quark model adopted here (with modified harmonic oscillator wavefunctions) and wavefunctions from potential models. Here we focus on calculations using modified harmonic oscillator wavefunctions, and we compare these with results obtained from other approaches.

IV. ANALYSIS OF RADIATIVE TRANSITIONS OF $\chi_{c1}(1P)$ AND $\chi_{b1}(nP)$

In this section we apply the light-front formalism for

the decay $A(1^{++}) \rightarrow V(1^{--}) + \gamma$ to the analysis of the

TABLE II: Coefficients f_2 , f_4 , and f_5 for $\chi_{b1}(nP) \rightarrow \Upsilon(n'S)\gamma$ in covariant light-front approach, where $f_i \equiv f_i(q^2 = 0)$.

Decay mode	f_2	$f_4(\text{GeV}^{-2})$	$f_5(\text{GeV}^{-2})$
$\chi_{b1}(1P) \rightarrow \Upsilon(1S)\gamma$	-0.94 ± 0.06	0.0049 ± 0.0004	-0.0083 ± 0.0002
$\chi_{b1}(2P) \rightarrow \Upsilon(1S)\gamma$	$+0.21 \pm 0.05$	0.0019 ± 0.0006	0.0037 ± 0.0002
$\chi_{b1}(2P) \rightarrow \Upsilon(2S)\gamma$	-1.26 ± 0.10	0.0094 ± 0.0010	-0.0071 ± 0.0008
$\chi_{b1}(3P) \rightarrow \Upsilon(1S)\gamma$	-0.11 ± 0.03	-0.0014 ± 0.0002	-0.0021 ± 0.0003
$\chi_{b1}(3P) \rightarrow \Upsilon(2S)\gamma$	$+0.29 \pm 0.10$	0.0038 ± 0.0016	0.0050 ± 0.0002
$\chi_{b1}(3P) \rightarrow \Upsilon(3S)\gamma$	-1.39 ± 0.06	0.0056 ± 0.0032	-0.0087 ± 0.0015

TABLE III: Decay widths (in units of keV) of $\chi_{b1}(nP) \rightarrow \Upsilon(n'S) + \gamma$ E1 decays in the light-front approach, denoted Γ_{MSHO} , based on modified simple harmonic oscillator (MSHO) wavefunctions [48]. The predictions from other models (relativistic quark model[29, 31], non-relativistic screened potential model [30], and nonrelativistic constituent quark model [32]) are also listed for comparison, where $[30]_0$ denotes results from the SNR_0 (screened nonrelativistic) model and $[30]_1$ denotes results from the SNR_1 model. We also list the ratio $\Gamma_{\text{MSHO}}/\Gamma_{\text{th(ave.)}}$, where $\Gamma_{\text{th(ave.)}}$ is average value of widths from other theoretical models.

Decay mode	Γ_{MSHO}	[29]	[30] ₀	[30] ₁	[31]	[32]	$\Gamma_{\text{MSHO}}/\Gamma_{\text{th(ave.)}}$
$\chi_{b1}(1P) \rightarrow \Upsilon(1S)\gamma$	37.3 ± 4.8	36.6	33.6	30.0	29.5	35.66	1.12 ± 0.15
$\chi_{b1}(2P) \rightarrow \Upsilon(1S)\gamma$	10.6 ± 5.5	7.49	12.4	8.56	5.5	9.13	1.23 ± 0.64
$\chi_{b1}(2P) \rightarrow \Upsilon(2S)\gamma$	10.0 ± 1.7	14.7	15.9	13.8	13.3	15.89	0.68 ± 0.12
$\chi_{b1}(3P) \rightarrow \Upsilon(1S)\gamma$	6.1 ± 3.9		6.80	3.39	1.3	4.17	1.56 ± 1.00
$\chi_{b1}(3P) \rightarrow \Upsilon(2S)\gamma$	4.7 ± 3.2		5.48	5.39	3.1	4.58	1.01 ± 0.69
$\chi_{b1}(3P) \rightarrow \Upsilon(3S)\gamma$	3.6 ± 0.4		12.0	9.97	8.4	9.62	0.36 ± 0.04

radiative decays $\chi_{c1}(1P) \rightarrow J/\psi + \gamma$ and $\chi_{b1}(nP) \rightarrow \Upsilon(n'S) + \gamma$. We present the results of our numerical calculations of form factors (evaluated at $q^2 = 0$) and decay widths. For the charmonium $\chi_{c1}(1P)$ decay, we compare our result with experimental data on the width, as listed in the Particle Data Group Review of Particle Properties (RPP) [71]. Although the RPP lists this width for the decay $\chi_{c1}(1P) \rightarrow J/\psi$, it does not list widths for the $\chi_{b1}(nP) \rightarrow \Upsilon(n'S) + \gamma$ decays, only branching ratios. Since our calculation yields the width itself, and a calculation of the branching ratio requires division by the total width in each case, we therefore compare our results on the branching ratios for these decays with predictions from other models, including the relativistic quark model [29, 31], the non-relativistic screened potential model [30], and the nonrelativistic constituent quark model [32]. For each decay, we have performed numerical calculations based on modified harmonic oscillator wavefunctions as discussed in [48].

First, we study the charmonium radiative decay $\chi_{c1}(1P) \rightarrow J/\psi + \gamma$. The parameter sets that we use are as follows, with labels indicated:

1. CM1: $m_c = 1.4 \text{ GeV}$,
 $\beta_{\chi_{c1}(J/\psi)} = 0.639 \pm 0.020 \text{ GeV}$.
2. CM2: $m_c = 1.5 \text{ GeV}$,
 $\beta_{\chi_{c1}(J/\psi)} = 0.600 \pm 0.020 \text{ GeV}$.

We present our results in Table I, with the uncertainties arising from the uncertainties in the β parameters, as in [49]. As one can see from Table I, our results agree

with experimental data within the range of experimental and theoretical uncertainties. The theoretical uncertainties arise from the value of m_c taken and also from the model used. The model-dependent uncertainties will be evident from our comparison of predictions from various models.

Next, we proceed to analyze the radiative decays of P-wave $b\bar{b}$ states. We use the modified harmonic oscillator wavefunctions, which have been successfully applied to the study of radiative decays of $\Upsilon(nS) \rightarrow \eta_b + \gamma$ [48]. In this case, the LFQM has the following parameters: the mass of the quark, m_b , the harmonic oscillator wavefunction parameter for $\chi_{b1}(nP)$, $\beta_{\chi_{b1}(nP)}$, and the wavefunction parameter for $\Upsilon(nS)$, $\beta_{\Upsilon(nS)}$. For the mass of the quark, we use $m_b = 4.8 \text{ GeV}$. This is an effective b -quark mass chosen to optimize the fit to these radiative transitions, as has been done in a number of other studies; for example, the recent comprehensive study [31] uses the value $m_b = 4.977 \text{ GeV}$.

For the effective harmonic oscillator wavefunction parameters, there are two choices. One is to use a single parameter β for all states in the $b\bar{b}$ system. In this case, the wavefunctions correspond to eigenstates of the harmonic oscillator Hamilton with $V \propto r^2$, and hence the energy splitting between different energy levels is [5] $\Delta E \propto \mu^{-1/2}$, where $\mu = m_Q/2$ is the reduced mass of $Q\bar{Q}$ system. This does not account for the observed approximate equality of mass splittings $m(\psi(2S)) - m(J/\psi) \simeq m(\Upsilon(2S)) - m(\Upsilon(1S))$. Therefore, a more practical choice is to treat β as variational

parameter to fit each state separately. For example, in Ref. [31], the authors obtain β by equating the rms radius of the harmonic oscillator wavefunction for the specified states with the rms radius of the wavefunctions calculated using the relativized quark model. Explicitly, for $n = 1$, $\beta \sim 0.9 - 1.2$ GeV, for $n = 2$, $\beta \sim 0.7 - 0.8$ GeV, and for $n = 3$, $\beta \sim 0.6 - 0.7$ GeV. For our modified harmonic wavefunctions, these results are not exact, but can serve as an estimate of the range of wavefunction parameters. In our analysis, we use the following values of wavefunction parameters:

1. $\beta_{\chi_{b1}(1P)} = 1.00 \pm 0.02$ GeV,
2. $\beta_{\chi_{b1}(2P)} = 0.71 \pm 0.02$ GeV,
3. $\beta_{\chi_{b1}(3P)} = 0.70 \pm 0.02$ GeV,
4. $\beta_{\Upsilon(1S)} = 0.90 \pm 0.02$ GeV,
5. $\beta_{\Upsilon(2S)} = 0.71 \pm 0.02$ GeV,
6. $\beta_{\Upsilon(3S)} = 0.70 \pm 0.02$ GeV.

Here we have used estimated values of the uncertainties in these parameters corresponding to those that we used in our previous study [49]. The uncertainties that we include with our resultant calculations of radiative decay widths incorporate these uncertainties.

For the values of form factors, we show typical results in Table II. The numerical results for the decay widths calculated with our parameter setting are listed in Table III. In both of these tables, we include the estimated uncertainties arising from the uncertainties in the input value of m_b and the input values of the β parameters. Since for $\chi_{b1}(nP)$ system, only branching ratios are experimentally determined, we compare our results, denoted Γ_{MSHO} , with those from other theoretical models. As an rough estimation, we define the average values of widths from these theoretical models [29–32] to be $\Gamma_{\text{th(ave.)}}$. It should be noted that for many of the decay modes, there is a substantial spread of values of branching ratios predicted by different models. We then calculate the ratio $\Gamma_{\text{MSHO}}/\Gamma_{\text{th(ave.)}}$ and list this ratio in Table II. The decay $\chi_{b1}(1P) \rightarrow \Upsilon(1S) + \gamma$ has a measured branching ratio $BR(\chi_{b1}(1P) \rightarrow \Upsilon(1S) + \gamma) = 33.9 \pm 2.2$ % [71]. For this decay mode, our predicted width agrees well with the average of the other models and, furthermore, the predictions of these other models agree well among themselves. The measured branching ratios for the radiative decays of the $\chi_{b1}(2P)$ are $BR(\chi_{b1}(2P) \rightarrow \Upsilon(2S) + \gamma) = 19.9 \pm 1.9$ % and $BR(\chi_{b1}(2P) \rightarrow \Upsilon(1S) + \gamma) = 9.2 \pm 0.8$ % [71]. Our predicted width for the first of these decays is in good agreement with the average of the predictions of other models, while our predicted width for the second of these decays is slightly smaller than this average. $\chi_{bJ}(3P)$ states have recently been observed at the LHC via their radiative decays [23, 24] (although no branching ratios for these decays are listed yet by the PDG). For radiative decays of

$\chi_{b1}(3P)$ to $\Upsilon(1S)$ and $\Upsilon(2S)$, our LFQM predictions are in good agreement, to within uncertainties, with other models, while our prediction for the decay to $\Upsilon(3S)$ is somewhat smaller than the predictions from other models.

In general, these results show that the light-front quark model with phenomenological meson wavefunctions (specifically, modified harmonic oscillator wavefunctions), is suitable for the calculation of $nP \rightarrow n'S$ radiative decay widths, since this model gives reasonable predictions for these widths, as compared with experimental data and other theoretical approaches. The results from the calculations in the covariant light-front approach and corresponding nonrelativistic/relativized quark model calculations reflect some differences in the predictions of decay widths, which are related to differences in the properties of these respective models. Specifically, nonrelativistic/relativized quark models contain different ways of including relativistic corrections and also truncations of these relativistic effects, while in the LFQM these relativistic corrections are systematically included. This shows one advantage of the covariant light-front approach, namely, that it is a fully relativistic formalism, and one does not need to carry out a reduction from relativistic interaction terms to the nonrelativistic limit.

One drawback in the current LFQM is that we do not know the exact form of the light-front wavefunctions and hence only use trial wavefunctions. This problem is more serious for excited states, because for excited $b\bar{b}$ states with radial quantum numbers $n \geq 2$, where Λ_{QCD} is larger than the typical binding energy of the state, the Coulombic type potential is no longer a very good approximation [13], so we have larger uncertainties in the $b\bar{b}$ wavefunction that serves as input in light-front quark model. This can be seen from Table III; for reasonable parameters, the decay width of $\chi_{b1}(1P)$ from the LFQM agrees well with predictions from nonrelativistic/relativized quark models, but for excited states, the LFQM calculations for two channels do not match perfectly with predictions from these nonrelativistic/relativized quark models. As been pointed out in Ref. [13], for radiative transition of these excited $b\bar{b}$ states, we rely on phenomenological models, but these do not always agree with QCD in the perturbative regime. Even though the LFQM is a fully relativistic approach, there is thus motivation for further theoretical work to gain a better understanding of the determination of light-front wavefunctions for QQ states.

V. CONCLUSION

In this paper we have derived formulas for the radiative decay of 1^{++} heavy mesons via the channel $1^{++} \rightarrow 1^{--} + \gamma$ in the light-front quark model. Then we have applied these to calculate the coefficients f_i and the radiative decay widths of $\chi_{c1}(1P)$ and $\chi_{b1}(nP)$

via the respective channels $\chi_{c1}(1P) \rightarrow J/\psi + \gamma$ and $\chi_{b1}(nP) \rightarrow \Upsilon(n'S) + \gamma$. Within the LFQM framework, we have adopted modified harmonic-oscillator wavefunctions. We have shown that most of the predictions of the LFQM with modified harmonic-oscillator wavefunctions are in reasonable agreement with data and other model calculations.

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Appendix A: Time reversal and Parity transformations of amplitude

1. Time Reversal Transformation

The action of time reversal on on S-matrix element $\langle\beta|H|\alpha\rangle$ is defined to be $(\langle\tilde{\beta}|\mathcal{T}H\mathcal{T}^{-1}|\tilde{\alpha}\rangle)^*$, where $|\tilde{\alpha}\rangle = \mathcal{T}|\alpha\rangle$. So the time-reversal invariance of electromagnetic interaction is [51]:

$$\langle\beta|H_{em}|\alpha\rangle = (\langle\tilde{\beta}|\mathcal{T}H_{em}\mathcal{T}^{-1}|\tilde{\alpha}\rangle)^* = (\langle\tilde{\beta}|H_{em}|\tilde{\alpha}\rangle)^* \quad (\text{A1})$$

where we use the time-reversal invariance of the electromagnetic Hamiltonian operator: $H_{em} = \mathcal{T}H_{em}\mathcal{T}^{-1}$.

For a state with 3-momentum \vec{p} , spin J and z -component of spin m , the time-reversal transformation is $\mathcal{T}|\vec{p}, J, m\rangle = \zeta(-1)^{J-m}|\vec{p}, J, -m\rangle$ (for vector and axial-vector states, $\zeta = +1$). After contractions with the associated field operator, this amounts to the change of polarization: $\varepsilon^\mu(\vec{p}, m) \rightarrow \zeta(-1)^{J-m}\varepsilon^\mu(-\vec{p}, -m) = \zeta(-1)^{J+1}P_\nu^\mu\varepsilon^{\nu*}(\vec{p}, m)$, where we have used the relation $\varepsilon^{\mu*}(\vec{p}, m) = (-1)^{m+1}P_\nu^\mu\varepsilon^\nu(-\vec{p}, -m)$, and $P_\nu^\mu = \text{diag}(1, -1, -1, -1)$ represents spatial inversion [51].

The general amplitude in Eq.(2.14) should satisfy the time-reversal invariance condition in Eq.A1. Let us consider the f_2 term first. Without loss of generality, we can choose polarization $(+, +, +)$ states; then this is given by

$$\mathcal{M}_{+++} = -i\varepsilon_\mu^*(q, +)\varepsilon'_\nu(P', +)\varepsilon''_k(P'', +)f_2\varepsilon^{\mu\nu\rho\alpha}q_\alpha. \quad (\text{A2})$$

In this case where the three polarization vectors are all transversal and only carry spatial components of Lorentz indices, the index of the photon momentum q_α has to be $\alpha = 0$:

$$\mathcal{M}_{+++} = -i\varepsilon_i^*(q, +)\varepsilon'_j(P', +)\varepsilon''_k(P'', +)f_2\varepsilon^{ijk}q_0. \quad (\text{A3})$$

Under a time-reversal transformation, $q^0 \rightarrow q^0$, $\varepsilon^i(\varepsilon'^i, \varepsilon''^i) \rightarrow (-1)\varepsilon^{i*}(\varepsilon'^{i*}, \varepsilon''^{i*})$ and

$$\begin{aligned} \widetilde{\mathcal{M}}_{+++} &= -(-1)^3(i\varepsilon_i(q, +)\varepsilon'_j(P', +)\varepsilon''_k(P'', +)f_2\varepsilon^{ijk}q_0)^* \\ &= -i\varepsilon_i^*(q, +)\varepsilon'_j(P', +)\varepsilon''_k(P'', +)f_2^*\varepsilon^{ijk}q_0. \quad (\text{A4}) \end{aligned}$$

According to Eq. (A1), the amplitude is time-reversal invariant if

$$\mathcal{M}_{+++} = \widetilde{\mathcal{M}}_{+++} \rightarrow f_2 = f_2^*, \quad (\text{A5})$$

which is satisfied as we can see from the explicit expression of f_2 in Eq.(2.29). Using an equivalent analysis, we can prove that the f_4 and f_5 terms also preserve time-reversal invariance.

2. Parity Transformation

For a physical state $|\alpha\rangle$, the action of a parity transformation is $\mathcal{P}|\alpha\rangle = |\alpha'\rangle$. The parity invariance of the electromagnetic interaction is expressed as

$$\langle\beta|H_{em}|\alpha\rangle = \langle\beta|\mathcal{P}^{-1}\mathcal{P}H_{em}\mathcal{P}^{-1}|\alpha\rangle = \langle\beta'|H_{em}|\alpha'\rangle \quad (\text{A6})$$

The parity transformation of a state with 3-momentum \vec{p} , spin J , and z -component of spin m is defined as $\mathcal{P}|\vec{p}, J, m\rangle = \eta_P|-\vec{p}, J, m\rangle$, where η_P is the intrinsic parity of this state. For a vector meson, $\eta_P = -1$, and for an axial vector meson, $\eta_P = +1$. After contractions with the associated field operator, this amounts to the change of polarization: $\varepsilon^\mu(\vec{p}, m) \rightarrow \eta_P\varepsilon^\mu(-\vec{p}, m) = -\eta_PP_\nu^\mu\varepsilon^\nu(\vec{p}, m)$, where we have used the relation $\varepsilon^\mu(-\vec{p}, m) = -P_\nu^\mu\varepsilon^\nu(\vec{p}, m)$.

The general amplitude in Eq.(2.14) should satisfy the parity invariance condition in Eq.A6. We take the f_2 term as an example to demonstrate this requirement. Without loss of generality, we can choose the polarization $(+, +, +)$ states, for which the amplitudes are given by Eq.(A2) and Eq.(A3).

Under a parity transformation, $q^0 \rightarrow q^0$, $\varepsilon^i(\varepsilon'^i, \varepsilon''^i) \rightarrow \eta_P(+1)\varepsilon^i(\varepsilon'^i, \varepsilon''^i)$, the amplitude \mathcal{M}_{+++} is transformed to

$$\begin{aligned} \mathcal{M}'_{+++} &= -\eta_V\eta_A\eta_\gamma i\varepsilon_i^*(q, +)\varepsilon'_j(P', +)\varepsilon''_k(P'', +)f_2\varepsilon^{ijk}q_0 \\ &= -(+1)i\varepsilon_i^*(q, +)\varepsilon'_j(P', +)\varepsilon''_k(P'', +)f_2\varepsilon^{ijk}q_0, \quad (\text{A7}) \end{aligned}$$

where the intrinsic parities of V , A and γ are $\eta_V = -1$, $\eta_A = +1$ and $\eta_\gamma = -1$, respectively. From Eq.(A7), we can see $\mathcal{M}'_{+++} = \mathcal{M}_{+++}$; hence parity is conserved for the f_2 term. Applying the same method of analysis, we can prove that the f_4 and f_5 terms also preserve parity invariance.

Appendix B: The wavefunctions

The normalization of the S-wave meson wavefunction in the light-front framework is

$$\frac{1}{2(2\pi)^3} \int dx_2 dp_\perp^2 |\phi_{n's}(x_2, p_\perp)|^2 = 1. \quad (\text{B1})$$

Here $\phi_{n's}(x_2, p_\perp)$ is related to the wavefunction in normal coordinates $\psi_{n's}(p)$ by

$$\phi_{n's}(x_2, p_\perp) = 4\pi^{\frac{3}{2}} \sqrt{\frac{dp_z}{dx_2}} \psi_{n's}(p) , \quad \frac{dp_z}{dx_2} = \frac{e'_1 e_2}{x_1 x_2 M'_0} \quad (\text{B2})$$

The normalization of $\psi_{n's}(p)$ is given by

$$\int d\mathbf{p}^3 |\psi_{n's}(p)|^2 = 4\pi \int p^2 dp |\psi_{n's}(p)|^2 = 1 . \quad (\text{B3})$$

The normalization for the P-wave meson wavefunction in the light-front framework is [44]

$$\frac{1}{2(2\pi)^3} \int dx_2 dp_\perp^2 |\phi_{np}(x_2, p_\perp)|^2 p_i p_j^* = \delta_{ij} , \quad (\text{B4})$$

where $p_i = (p^+, p^-, p_z)$. In terms of the P-wave wavefunction in normal coordinates,

$$\phi_{np}(x_2, p_\perp) = 4\pi^{\frac{3}{2}} \sqrt{\frac{dp_z}{dx_2}} \psi_p(p) , \quad \frac{dp_z}{dx_2} = \frac{e'_1 e_2}{x_1 x_2 M'_0} \quad (\text{B5})$$

we have the following normalization condition:

$$\frac{1}{3} \cdot 4\pi \int_0^\infty |\psi_{np}(p)|^2 p^4 dp = 1 . \quad (\text{B6})$$

For the gaussian type 1P and 1S wavefunctions, we have the relation

$$\psi_p^{1P}(p) = \sqrt{\frac{2}{\beta^2}} \psi_s^{1S}(p) . \quad (\text{B7})$$

The modified harmonic oscillator n-S wavefunctions in the light-front approach are [48]

$$\begin{aligned} \psi_{s,M}^{1S}(p) &= \left(\frac{1}{\beta^2 \pi} \right)^{\frac{3}{4}} \exp \left(-\frac{1}{2} \frac{p^2}{\beta^2} \right) \\ \psi_{s,M}^{2S}(p) &= \left(\frac{1}{\beta^2 \pi} \right)^{\frac{3}{4}} \exp \left(-\frac{2^\delta}{2} \frac{p^2}{\beta^2} \right) \left(a'_2 - b'_2 \frac{p^2}{\beta^2} \right) \\ \psi_{s,M}^{3S}(p) &= \left(\frac{1}{\beta^2 \pi} \right)^{\frac{3}{4}} \exp \left(-\frac{3^\delta}{2} \frac{p^2}{\beta^2} \right) \left(a'_3 - b'_3 \frac{p^2}{\beta^2} + c'_3 \frac{p^4}{\beta^4} \right) \end{aligned} \quad (\text{B8})$$

where

$$\begin{aligned} a'_2 &= 1.88684 & b'_2 &= 1.54943 \\ a'_3 &= 2.53764 & b'_3 &= 5.67431 & c'_3 &= 1.85652 \\ \delta &= 1/1.82 . \end{aligned} \quad (\text{B9})$$

The nP wavefunction, is related to nS wavefunction as follows:

$$\psi_{p,M}^{nP}(p) = \frac{C_n}{\beta} \psi_{s,M}^{nS}(p) , \quad (\text{B10})$$

where the constants C_n can be determined by the normalization condition in Eq.(B4) as

$$\begin{aligned} C_1 &= \sqrt{2} = 1.41421 \\ C_2 &= 1.23833 \\ C_3 &= 1.13215 . \end{aligned} \quad (\text{B11})$$

Appendix C: Some expressions in the light-front formalism

In the covariant light-front formalism we have

$$\begin{aligned} M_0'^2 &= (e'_1 + e_2)^2 = \frac{p_\perp'^2 + m_1'^2}{x_1} + \frac{p_\perp'^2 + m_2^2}{x_2} \\ M_0''^2 &= (e'_1 + e_2)^2 = \frac{p_\perp''^2 + m_1''^2}{x_1} + \frac{p_\perp''^2 + m_2^2}{x_2} \\ \tilde{M}'_0 &= \sqrt{M_0'^2 - (m'_1 - m_2)^2} \\ \tilde{M}''_0 &= \sqrt{M_0''^2 - (m''_1 - m_2)^2} \\ p'_z &= \frac{x_2 M'_0}{2} - \frac{m_2^2 + p_\perp'^2}{2x_2 M'_0} \\ p''_z &= \frac{x_2 M''_0}{2} - \frac{m_2^2 + p_\perp''^2}{2x_2 M''_0} \\ e'_1 &= \sqrt{m_1'^2 + p_\perp'^2 + p_z'^2} \\ e''_1 &= \sqrt{m_1''^2 + p_\perp''^2 + p_z''^2} \\ e_2 &= \sqrt{m_2^2 + p_\perp'^2 + p_z'^2} , \end{aligned} \quad (\text{C1})$$

The explicit expressions for $A_j^{(i)}(i, j = 1 \sim 4)$ are

$$\begin{aligned} A_1^{(1)} &= \frac{x_1}{2} , \quad A_2^{(1)} = A_1^{(1)} - \frac{p'_\perp \cdot q_\perp}{q^2} , \\ A_1^{(2)} &= -p_\perp'^2 - \frac{(p'_\perp \cdot q_\perp)^2}{q^2} , \quad A_2^{(2)} = (A_1^{(1)})^2 , \\ A_3^{(2)} &= A_1^{(1)} A_2^{(1)} , \quad A_4^{(2)} = (A_2^{(1)})^2 - \frac{1}{q^2} A_1^{(2)} , \\ A_1^{(3)} &= A_1^{(1)} A_1^{(2)} , \quad A_2^{(3)} = A_2^{(1)} A_1^{(2)} , \\ A_3^{(3)} &= A_1^{(1)} A_2^{(2)} , \quad A_4^{(3)} = A_2^{(1)} A_2^{(2)} . \end{aligned} \quad (\text{C2})$$

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